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Optimum spines with temperature dependent thermal parameters

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Abstract—Based on the principle of minimum volume, an optimum shape of spine is proposed at a given heat duty, base temperature, and fin length. Both the thermal conductivity and heat transfer coefficient are temperature dependent. In this study, a power-law type wall heat flux is employed. A uniform temperature is assumed at the fin base whereas two kinds of boundary at the fin tip are considered. First, a specified heat transfer rate is prescribed at the free end. Second, the outer edge of the fin is also subject to a power-law type surface heat flux and dissipates energy to the ambient fluids. Special cases of negligible heat transfer from fin tip and zero tip temperature are investigated. The temperature distributions and the profiles of the optimum fins are calculated for various single heat transfer modes. The results are presented in non-dimensionalized form for the convenience of parametric study and design analysis.

INTRODUCTION

EXTENDED surfaces are frequently used in heat exchange devices for the purpose of increasing the heat transfer between a primary surface and the surrounding fluid. In many applications, optimization of the fins is of great interest, such as in the design of cooling devices on vehicles, especially aircraft. The investigation of exchanging the greatest amount of heat with the least amount of weight in the exchanger is very important. For a fin with uniform material, this research leads to the consideration of the minimum volume.

The optimization problem lies either in finding the shape and dimension of the fins which would minimize the volume for a given amount of heat dissipation, or, alternatively, to maximize the heat dissipation for a given volume. For pure conducting fins, a criterion for optimum shape was proposed by Schmidt [1] using the principle of a constant heat flux. Later, Duffin [2] confirmed this result by applying a rigorous variational approach. In their studies, the fin profile is calculated to be a parabola and has a zero thickness at the outer edge. The effect of internal heat generation on the optimum shape was first considered by Minkler and Rouleau [3] but a more rigorous treatment was given by Liu [4] for heat generations which are directly proportional to the temperature. An optimum shape of a purely radiating fin was obtained by Wilkins [5–7] for a variety of geometries. Chung and Zhang [8, 9] dealt with the minimization of the mass of a radiative fin array for which the mutual irradiation between fin and base was taken into consideration. In addition, a great amount of literature to date has reported different aspects of the problem of fin optimization which have been investigated with many additional conditions [10–13]. However, none of them is applicable to boiling heat transfer.

In the use of fins in efficient heat exchangers, such as boiling heat transfer, a power-law type heat transfer coefficient is more representative of the real phenomena. In general, the heat transfer coefficient can be expressed as an empirical function of the temperature difference between the fin and liquid given by the "boiling curve". Haley and Westwater [14] used a numerical computation to find out an optimum shape of the spine fin which turns out to resemble a spade on a playing card. Yeh and Liaw [15] studied the same problem with a different design approach. The outer appearance of their optimum spine is just like a cylindrical fin, and its inner portion is nozzle-shaped. Employing a temperature correlated profile, a few authors [16, 17] investigated the optimization of fins with temperature dependent heat transfer coefficient. However, all of these works regarding boiling heat transfer consider negligible heat transfer from the free end, and thus their fin shapes have sharp edges at the fin tips. This causes an even more difficult manufacturing problem. Recently, Chung and Iyer [18] have used an integral approach and derived the optimum dimensions for longitudinal rectangular fins and cylindrical pin fins by incorporating transverse conduction.

In practical applications, the size of a heat exchanger may not be so large due to space considerations. Thus, the fin length must be restricted. In this study, based on the principle of minimum volume and with the given conditions of heat duty, base temperature, and fin length, an optimum spine with heat transfer from the free end is proposed. The temperature dependent thermal conductivity and heat transfer coefficient are taken into account. By inte-

NOMENCLATURE

a	dimensional constant related to a selected heat transfer mode $[\mathrm{W\,m^{-2}\,K^{-m}}]$
Bi	Biot number, $h_b \cdot L/k$
c_1, c_2, c_3	constants
h	heat transfer coefficient $[\mathrm{W\,m^{-2}\,K^{-1}}]$
k	thermal conductivity $[\mathrm{W\,m^{-1}\,K^{-1}}]$
L	fin length [m]
m	power-law exponent
Q	total heat transfer of a fin [W]
q	surface heat flux of fin $[\mathrm{W\,m^{-2}}]$
r	radius of spine [m]
$\tilde{r}$	dimensionless radius, r/L_r
T	temperature [K]
u, v	transformation parameters defined in equations (5) and (6)
V	volume of fin $[\mathrm{m^3}]$
$\tilde{V}$	dimensionless volume, V/V_r
z	coordinate in axial direction
$\tilde{z}$	dimensionless fin length, z/L_r
ΔT	temperature superheat, $T - T_a$ [K].

Greek symbols

α	variable defined in equation (21)
β	variable defined in equation (27)
ε	fin effectiveness
γ	parameter defined in Appendix 1
λ	variable defined in equation (26)
θ	dimensionless temperature.

Subscripts and superscripts

a	ambient
b	fin base
crit	critical
min	minimum
r	reference
o	fin tip
*	non-dimensional quantity.

grating the non-linear fin equation analytically, the temperature distributions of the fins are obtained in the form of hypergeometric functions. The calculated fin profiles are compared with previous work. In addition, various single heat transfer modes are investigated by choosing appropriate power-law exponents.

THEORETICAL ANALYSIS

An axi-symmetric fin with a temperature dependent thermal conductivity is cooled by the ambient fluid. The heat is conducted through the fin material and finally dissipated to the surroundings also with a temperature dependent heat transfer coefficient, which is a function of the temperature difference between wall and the ambient fluid. No heat source or sink exists within the fin. The ambient fluid temperature is maintained at a constant temperature, T_a, and a uniform fin base temperature of T_b is assumed. The distance from the spine axis to any point on its surface is denoted by $r(z)$.

Before solving the problem analytically, it is convenient to non-dimensionalize the governing equation and boundary conditions. Selecting the following group of dimensionless variables:

$$Q^* = \frac{Q}{\pi L k(T)\,\Delta T_b} \quad z^* = \frac{z}{L}$$

$$r^* = \frac{r}{L} \quad \theta = \frac{\Delta T}{\Delta T_b}$$

$$q^* = \frac{Lq}{\Delta T_b\, k(T)},$$

the one-dimensional steady-state heat conduction equation inside the fin expressed in dimensionless form is:

$$\frac{dQ^*}{dz^*} = \frac{d}{dz^*}\left(r^{*2}\frac{d\theta}{dz^*}\right) = 2r^*q^*. \qquad (1)$$

Note that the dissipation function, q^*, is both temperature and position dependent, i.e. $q^* = q^*(z^*, \theta)$, which is known for any specified problem. Considering that z^* is a monotonic function of θ, it may be written $q^* = q^*[z^*(\theta), \theta] = q^*(\theta)$. Figure 1 shows the physical model and coordinate system for a spine with an arbitrary profile. The origin of the coordinate system is taken at the center of the spine tip. With a uniform temperature at the fin base and the imposition of a given transfer rate at the tip, the boundary conditions are written as:

$$Q^* = Q_0^* \quad \text{at} \quad z^* = 0, \qquad (2)$$

$$Q^* = Q_b^* \quad \theta = 1 \quad \text{at} \quad z^* = 1. \qquad (3)$$

If the fin tip surface is subject to a power-law wall heat flux, the boundary condition (2) then becomes:

$$Q^* = r_0^{*2} q_0^* \quad \text{at} \quad z^* = 0. \qquad (4)$$

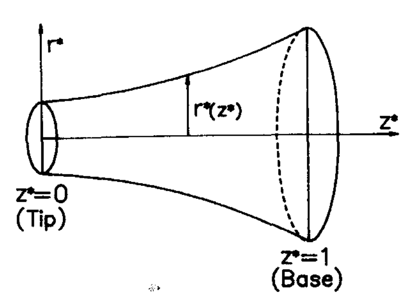

FIG. 1. Coordinate system for the proposed spine.

To simplify the optimization procedure, the transformation parameters [7] are employed and modified as follows:

$$u = \int_{\theta_0}^{\theta} q^{*4}\,\mathrm{d}\theta, \tag{5}$$

$$v = \left[\frac{Q^*}{Q_b^*}\right]^{5/4}. \tag{6}$$

The use of the transformation parameters in the fin optimization problem can also be found in many works [9, 15, 19].

Differentiating equation (6) with respect to u, gives:

$$\mathrm{d}Q^* = \frac{4Q_b^*}{5} v^{-1/5}\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)\mathrm{d}u. \tag{7}$$

Substituting into equation (1) yields:

$$\mathrm{d}z^* = \left(\frac{4Q_b^*}{25}\right)^{1/3} v^{-2/5}\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^{2/3} q^{*-2}\,\mathrm{d}u. \tag{8}$$

Employing equations (1), (5) and (6), the above equation becomes:

$$r^{*2} = \left(\frac{4Q_b^{*4}}{25}\right)^{1/3} v^{2/5}\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^{2/3} q^{*2}. \tag{9}$$

A dimensionless fin volume, V^*, is derived as:

$$V^* = \int_0^1 r^{*2}\,\mathrm{d}z^*. \tag{10}$$

In the following derivation, the volume of a fin is minimized. With the aid of equations (8) and (9) and after some manipulations, it takes the form:

$$V^* = \left(\frac{16Q_b^{*5}}{625}\right)^{1/3}\int_0^{u_b}\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^{4/3}\mathrm{d}u, \tag{11}$$

where u_b is an integration of equation (5) from the superheat of fin tip ($\theta = \theta_0$) to fin base ($\theta = 1$). From the theory of variational calculus, the minimum volume, V^*, of the proposed fin can be obtained by solving the Euler–Lagrange equation. It yields:

$$v = c_1 u + c_2. \tag{12}$$

Imposing the boundary conditions (2) and (3) and with the aid of the equations (5) and (6), equation (12) becomes:

$$v = (1-v_0)\frac{u}{u_b} + v_0. \tag{13}$$

From equations (5) and (8), the relation between the local position and temperature of the fin can be calculated as:

$$z^* = \left[\frac{4Q_b^*(1-v_0)^2}{25u_b{}^2}\right]^{1/3} \times \int_{\theta_0}^{\theta}\left[(1-v_0)\frac{u}{u_b}+v_0\right]^{-2/5} q^{*2}\,\mathrm{d}\theta. \tag{14}$$

Accordingly, from equation (9), the radius of the optimum fin along the axis can be expressed as:

$$r^* = \left[\frac{2Q_b^{*2}(1-v_0)}{5u_b}\right]^{1/3}\left[\left(1-v_0\right)\frac{u}{u_b}+v_0\right]^{1/5} q^*. \tag{15}$$

The principle of this optimization work is to determine the least volume of a fin to dissipate a given heat duty at a given base temperature and fin length. In this study, two cases are investigated. First a specified heat transfer rate is prescribed at the fin tip. Thus, for a known Q_b^* and v_0, the tip temperature, θ_0, can be calculated from equation (14) with the imposition of boundary condition (3). Second the outer edge of the fin is in contact with ambient fluid. Energy is transferred from the fin tip to the environmental fluid for which the heat transfer coefficient behaves the same as that around the lateral surface of the fin. In this case, v_0 and θ_0 can be obtained by solving equations (6) and (14) under boundary conditions (4) and (3) simultaneously at a given Q_b^*. The minimum volume of the fin is rewritten in the form:

$$V_{\min}^* = \left(\frac{16Q_b^{*5}}{625u_b}\right)^{1/3}(1-v_0)^{3/4}. \tag{16}$$

Usually, the surface heat flux, q^*, is dependent upon heat transfer modes. In the use of boiling from fins, q^* is assumed to be attainable from a boiling curve since many heat transfer modes may coexist on a fin. Once Q_b^* is prescribed, the temperature distribution can be obtained. In addition, the least volume as well as the geometry of the proposed fin can be determined.

Fin effectiveness, ε, is defined as the ratio of the fin heat transfer rate to the heat transfer rate dissipated without the fin. The fin effectiveness of the optimum spine is derived as:

$$\varepsilon = \left[\frac{25u_b^2}{4(1-v_0)^2 Q_b^*}\right]^{1/3} \Big/ q_b^{*3}. \tag{17}$$

FIN SUBJECT TO A POWER-LAW WALL HEAT FLUX

This optimization problem can be simplified if the wall heat flux can be approximated by a power dependence, i.e. a temperature dependent heat transfer coefficient is assumed. The expression of the heat transfer function may then be written as:

$$q^* = Bi\,\theta^m k^{*-1}, \tag{18}$$

where Bi is equal to $h_b \cdot L/k_b$ and h_b represents the heat transfer coefficient at the fin base, i.e. $h_b = a\Delta T_b^{m-1}$. The dimensionless thermal conductivity is defined as:

$$k^* = \frac{k(T)}{k_b} = \sum_n K_n \theta^n, \tag{19}$$

where K_n are usually obtained from curve fitting of experimental data. In some situations, a single heat transfer mode can be represented by an appropriate choice of a and m. For instance, the exponent m may take the values of 0.75, 1, 1.25, 3, and 4 when the fin is cooled due to film boiling, forced convection, free convection, nucleate boiling, and radiation into free space at zero temperature, respectively.

The temperature distribution can be derived from equation (14) and has the following form:

$$z^* = \left[\frac{4Q_b^*(1-v_0)^2}{25Bi^2\alpha_b^2}\right]^{1/3} \times \int_{\theta_0}^{\theta}\left[(1-v_0)\frac{\alpha}{\alpha_b}+v_0\right]^{-2/5}\theta^{2m}k^{*-2}\,d\theta, \quad (20)$$

where

$$\alpha = \int_{\theta_0}^{\theta}\theta^{4m}k^{*-4}\,d\theta. \quad (21)$$

The shape of the optimum spine becomes:

$$r^* = \left[\frac{2Q_b^{*2}(1-v_0)}{5Bi\,\alpha_b}\right]^{1/3}\left[(1-v_0)\frac{\alpha}{\alpha_b}+v_0\right]^{1/5}\theta^m k^{*-1}. \quad (22)$$

When a given heat transfer rate is prescribed at the fin tip, θ_0 can be evaluated by imposing boundary condition (3) on equation (20). As the free end dissipates heat to the ambient fluid, the heat transfer from the tip can be obtained with the aid of equations (4), (6) and (15) and is written as:

$$v_0^{2/5} = \left[\frac{4Q_b^*Bi(1-v_0)^2}{25\alpha_b^2}\right]^{1/3}\theta_0^{3m}k_0^{*-3}. \quad (23)$$

In this case, the two unknowns, v_0 and θ_0, can be determined from non-linear equation (19), at base condition, and equation (23). The two-equation Newton–Raphson method is used to find the roots. Since both the values of v_0 and θ_0 fall in the range between 0 and 1, the initial guesses can always be given sufficiently close to the exact solutions. This is helpful in the assurances as well as acceleration of convergence.

The expressions above can be further simplified if constant thermal conductivity is assumed, i.e. $k^* = 1$. In this case, as $v_0 = 0$ and $\theta_0 = 0$, using boundary condition (3), equation (20) may be reduced in the form:

$$\theta = z^{*5/(2m+3)}. \quad (24)$$

For $v_0 \neq 0$ and $\theta_0 \neq 0$, one may transform equation (20) into an incomplete beta function [20] which can also be expressed as:

$$z^* = \frac{1}{3}\left[\frac{20Q_b^*}{Bi^2(4m+1)}\right]^{1/3}\lambda^{4/15}\beta^{(2m+3)/5} \times\left\{[1-(\beta/\theta)^{4m+1}]^{3/5}F\left[\frac{3}{5},\frac{22m+8}{5(4m+1)};\frac{8}{5};1-(\beta/\theta)^{4m+1}\right]-c_3\right\}, \quad (25)$$

where:

$$\lambda = \frac{1-v_0}{1-\theta_0^{4m+1}}, \quad (26)$$

$$\beta = \left(\theta_0^{4m+1}-\frac{v_0}{\lambda}\right)^{1/(4m+1)}, \quad (27)$$

and:

$$c_3 = \left(\frac{v_0}{\lambda\theta_0^{4m+1}}\right)^{3/5}F\left[\frac{3}{5},\frac{22m+8}{5(4m+1)};\frac{8}{5};\frac{v_0}{\lambda\theta_0^{4m+1}}\right]. \quad (28)$$

In the above equations, F is a hypergeometric function which may be expanded in a form of asymptotic series. In the case of $v_0 = 0$ and $\theta_0 \neq 0$, c_3 vanishes and β is equal to θ_0. The solution method is identical to that described in the previous case. In addition, it is noted that the tip temperatures can be expressed explicitly by v_0 for $m = 0$, 0.2, and 2 as can be seen from equation (23). The results are given in Appendix 1. In these special cases, θ_0 can be directly substituted into equation (25) to solve v_0. Since θ_0 is eliminated and merely one unknown is handled, the solution method is apparently much easier.

The configuration and the volume of the optimum fin is described by:

$$r^* = \left[\frac{2(4m+1)Q_b^{*2}\lambda}{5Bi}\right]^{1/3}\cdot[\lambda(\theta^{4m+1}-\theta_0^{4m+1})+v_0]^{1/5}\theta^m, \quad (29)$$

and:

$$V_{\min}^* = \left[\frac{16(4m+1)Q_b^{*5}\lambda}{625Bi^4}\right]^{1/3}(1-v_0). \quad (30)$$

The effectiveness of the fin can be evaluated as:

$$\varepsilon = \left[\frac{25\lambda^2}{4(4m+1)^2Q_b^*Bi}\right]^{1/3}. \quad (31)$$

Also, in the case of $v_0 = 0$ and $\theta_0 = 0$, r^*, $V_{\min}^*$ and ε become:

$$r^* = \frac{(2m+3)^2Bi}{10(4m+1)}z^{*(9m+1)/(2m+3)}, \quad (32)$$

$$V_{\min}^* = 2\times10^{-3}\frac{(2m+3)^5Bi^2}{(4m+1)^3}, \quad (33)$$

and:

$$\varepsilon = \frac{5}{(2m+3)Bi}. \quad (34)$$

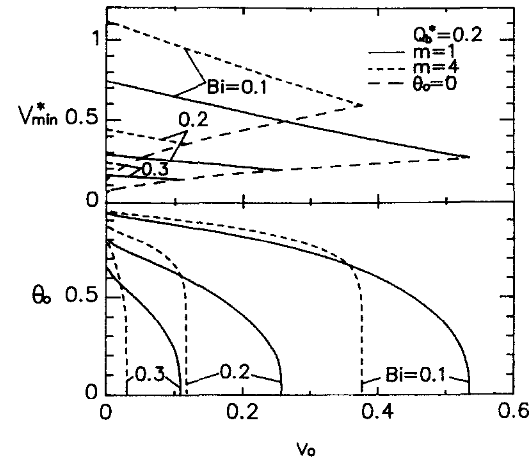

FIG. 2. Minimum volume and the corresponding tip temperature of the optimum spine with a fixed heat transfer rate prescribed at the free end surface.

RESULTS AND DISCUSSION

An optimum spine is theoretically obtained when the thermal conductivity and heat transfer coefficient are functions of temperature. For brevity, the assumptions of the power-law type heat transfer coefficient and a constant thermal conductivity for fin material is employed in this section. Initially, a given heat transfer rate prescribed at tip surface is discussed. Figure 2 shows the minimum volume of the optimum spine with the corresponding tip temperature. As the removal of heat, v_0, from the tip increases, the requirement of the dimensionless fin volume decreases at a given Q_b^*. The minimum volume of the optimum fin reaches the lowest limit when the tip temperature is equal to ambient fluid temperature. This can be observed from the lower half of Fig. 2. In this case, the optimum fin fails to exist for further increase of v_0 at the given conditions. Figure 3 depicts the fin profiles and temperature distributions along the fin length at $v_0 = 0.035$. Due to symmetry, only the upper half of the fin is displayed. For a fin with a smaller radius at the free end, the temperature distribution is concave downward for $m = 4$. The result is similar to the works of Haley and Westwater [14]. It is noted that a larger r_b^* is needed to achieve a larger Q_b^*. Thus, in this study, the use of a smaller Q_b^* is suggested for accuracy.

Special attention is given on the tip temperatures for $v_0 = 0$ in Fig. 2. When there is no heat transfer from the fin tip, the tip temperature does not equal zero ($\theta_0 \neq 0$). This has been a controversial topic among several investigators. An optimum fin profile with a zero thickness at the free end and a temperature differing from that of the surrounding environment was proposed by Mikk [12] for $m = 1$ and by Sohrabpour and Razani [17] for $7 > m > 1$. However, Schmidt [1], Duffin [2], and Sohrabpour and Razani [17] pointed out that the optimum fin profile is given by a parabola with vanishing thickness at the fin tip and $\theta_0 = 0$ for uniform heat transfer coefficient ($m = 1$). The dependence of θ_0 on Bi for $m = 1$ and $m = 4$ is given in Fig. 4. It is seen that the tip temperature may or may not drop to zero depending on the axial Biot number, i.e. the length of the optimum fin. From equation (20), the critical Bi for $v_0 = 0$ and $\theta_0 = 0$ is exactly obtained as:

$$(Bi)_{\text{crit}} = 2(4m+1)\left[\frac{5Q_b^*}{(2m+3)^3}\right]^{1/2}. \qquad (35)$$

The above expressions can also be obtained from equation (25). Details of the mathematical approach are given in Appendix 2. In addition to $m = 1$ and $m = 4$, similar trends are found for all other ms. The comparisons of the proposed fin profiles ($\theta_0 \neq 0$) and the profiles ($\theta_0 = 0$) of previous work [14] for $Bi = 0.23$, 0.25, and 0.27 are displayed in Fig. 5. It is observed that both fins have sharp tips. The two profiles become slenderer and tend to be identical as Bi

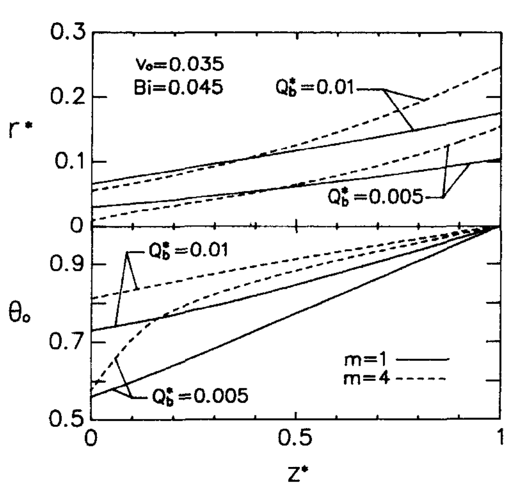

FIG. 3. Optimum fin profiles with the corresponding temperature distributions for $v_0 = 0.035$ and $Bi = 0.045$ ($m = 1$ and 4).

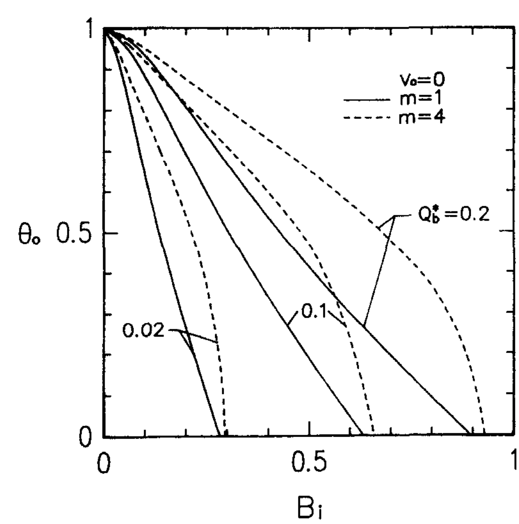

FIG. 4. Dependence of θ_0 on Bi for $m = 1$ and 4 and $Q_b^* = 0.02$, 0.1, and 0.2 ($v_0 = 0$).

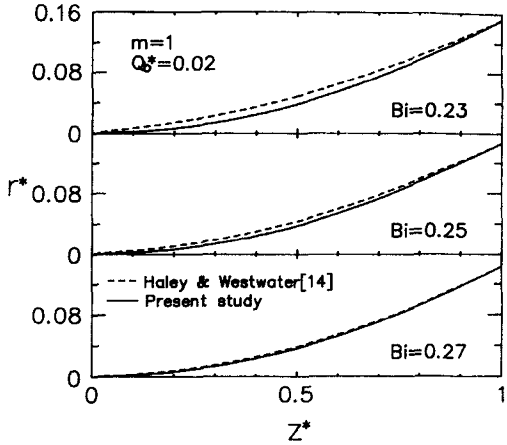

FIG. 5. Variations of radii along the length of the spines for $m = 1$ and $Bi = 0.23$, 0.25, and 0.27.

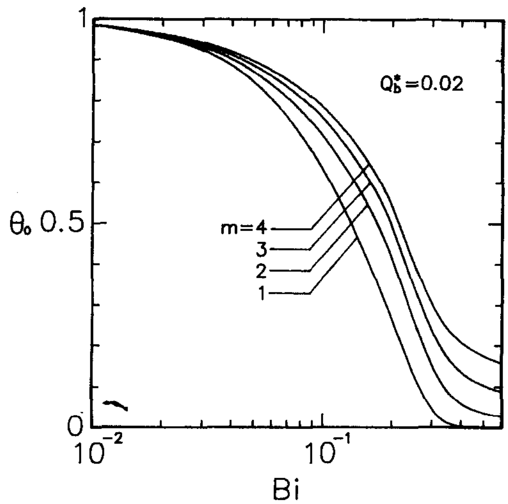

FIG. 6. Tip temperature of the optimum spines with the free end surfaces subject to power-law type wall heat flux.

increases. In addition, by substituting $v_0 = 0$ into equation (29), it may also be calculated that the optimum fin profiles have sharp edges.

When the tip surface heat transfer into the ambient fluid and the heat transfer coefficient on the free end of a fin are the same as those on the lateral surfaces, the optimum fin profiles are mainly determined by Q_b^* and Bi. Figure 6 shows the dependence of θ_0 on Bi for $Q_b^* = 0.02$. It is shown that the tip temperature increases with m at fixed Bi; however, no significant difference in θ_0 is observed for all the exponents of m at smaller Bi. Besides, θ_0 decreases as Bi increases. Thus, conceivably, v_0 tends to zero for a larger Bi. The shapes and temperature distributions of the optimum fin for $Bi = 0.6$ are given in Fig. 7. The tip of the proposed fin is sharper for a larger m. This is due to the fact that the heat duty is fixed and all the energy is dissipated through the periphery of a long spine. This result is also shown by Laor and Kalman [21] for non-zero tip temperature on an optimum spine with zero tip area.

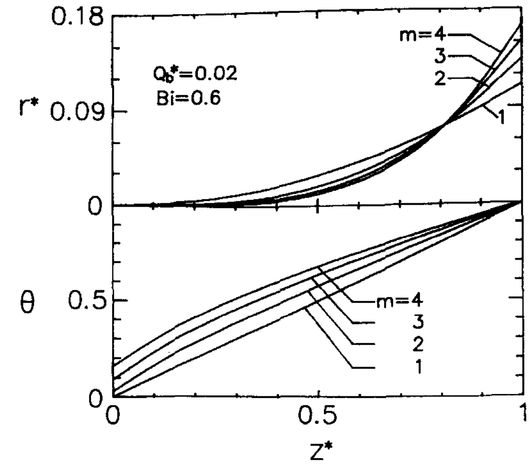

FIG. 7. The shapes and temperature distributions of the optimum spines with tip surfaces subject to power-law type wall heat flux.

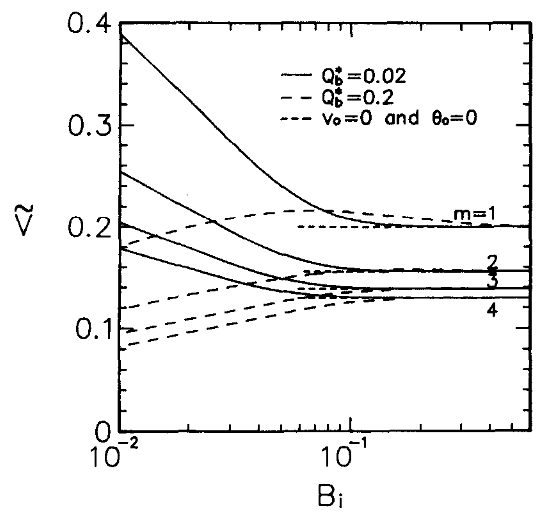

FIG. 8. Dependence of $\tilde{V}$ on Bi for $Q_b^* = 0.02$ and 0.2 and $m = 1$, 2, 3, and 4.

Since the length of the optimum fin varies with the given conditions, it is hard to directly tell which is smaller from the dimensionless fin profiles or volumes. Thus, the volume of a cylindrical fin with the length, L_r, and radius (at base) of the optimum spine for $v_0 = 0$ and $\theta_0 = 0$ is selected as a reference volume V_r. A new dimensionless volume, $\tilde{V}$, is defined by dividing the volume of the optimum spine by the reference volume. It is derived as:

$$\tilde{V} = \frac{2m+3}{5(4m+1)}\left[\frac{(1-v_0)^4}{1-\theta_0^{4m+1}}\right]^{1/3}. \quad (36)$$

It is interesting to note that $\tilde{V}$ is determined by m only for zero tip temperature and no heat transfer from fin tip. The dependence of $\tilde{V}$ on Bi for $Q_b^* = 0.02$ and 0.2 is presented in Fig. 8. It is shown that as Bi increases $\tilde{V}$ first increases to a maximum value then decreases to a constant. The maximum $\tilde{V}$ becomes larger and occurs at a smaller Bi for a smaller Q_b^*. Note that the

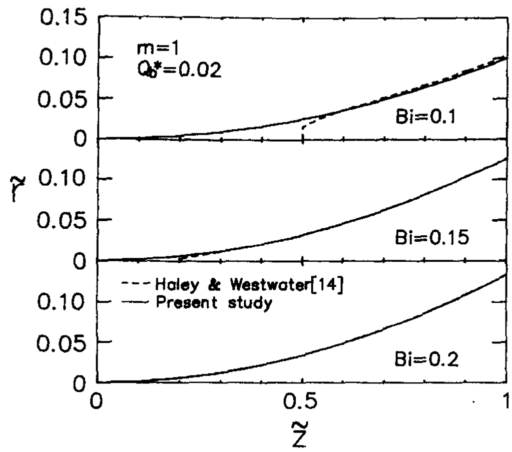

FIG. 9. Variations of radii along the length of the spines for $m = 1$ and $Bi = 0.1, 0.15$, and 0.2.

maximum $\tilde{V}$ for $Q_b^* = 0.02$ is not displayed in the figure. However, this phenomenon is not pronounced for a larger m. For instance, when $m = 4$ and $Q_b^* = 0.2$, $\tilde{V}$ only increases with Bi and reaches a fixed value evaluated from equation (36) with $v_0 = 0$ and $\theta_0 = 0$ at a larger Bi. Hence, the effect of heat transfer from tip surface is important for radiation. Similar results are also found in nucleate boiling.

In order to acquire a good understanding of the influence of Bi upon $\tilde{V}$, the optimum fin profiles for selected Bi are investigated. The variations of radii along the length of optimum spines for $Bi = 0.1, 0.15$, and 0.2 are plotted in Fig. 9. For clarity, the reference length L_r is used and all the fin profiles are shifted to the right end, i.e. all the fin bases are positioned at $z/L_r = 1$. These figures show that as Bi increases the fin profile changes from a thin truncated conical spine to a spine with a sharp tip. In addition, the two fin profiles tend to coincide at $Bi = 0.2$. Figure 10 illustrates the fin effectiveness of the proposed fin for

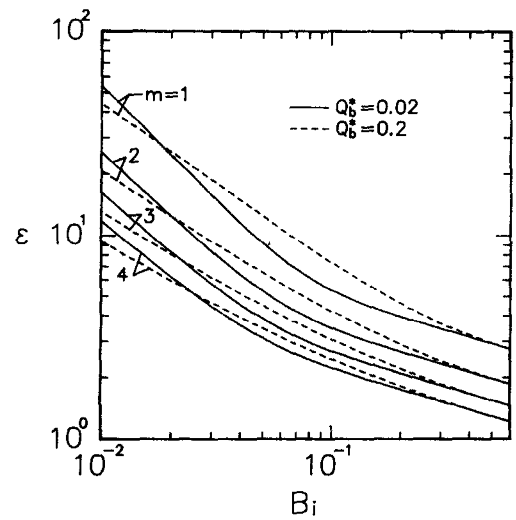

FIG. 10. Fin effectiveness of the optimum spines for $Q_b^* = 0.02$ and 0.2.

$Q_b^* = 0.02$ and 0.2. It shows that ε decreases as Bi increases. In addition, Q_b^* has little effect on ε for $Bi > 0.6$.

CONCLUSIONS

In this study, the optimum spines with minimum volume are obtained at a given Q_b^* and Bi. The main conclusions of the results are as follows:

(1) For a heat transfer rate prescribed at the surface of the fin tip, V_{min}^* decreases with θ_0 and the smallest V_{min}^* occurs at $\theta_0 = 0$.
(2) For the optimum fin with sharp tip ($v_0 = 0$), the tip temperature of the optimum fin may or may not equal the ambient fluid temperature depending upon the axial Biot number.
(3) When the free end of the optimum fin transfers heat by a power-type cooling process, the required fin volume increases with Bi to a maximum then decreases to a constant value. This phenomenon is more pronounced for a smaller Q_b^*.
(4) For a given heat duty, as the fin length increases the optimum fin profile changes from a truncated conical spine to a spine with vanishing radius at its tip.

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APPENDIX 1

For $m = 0$, the tip temperature can be expressed as a function of v_0, Q_b^*, and Bi. After some algebraic manipulation, equation (23) reduces to:

$$\theta_0 = 1 - 0.4\gamma, \tag{A1}$$

where:

$$\gamma = (1-v_0)v_0^{-3/5}\sqrt{Q_b^* Bi}. \tag{A2}$$

Similarly, for $m = 0.2$, θ_0 is derived as:

$$\theta_0 = (\sqrt{0.1296\gamma^2+1} - 0.36\gamma)^{1/0.9}, \tag{A3}$$

and, for $m = 2$, one obtains:

$$\theta_0 = \left(\frac{1}{1+3.6\gamma}\right)^{1/9}, \tag{A4}$$

where γ is identical to the expression given in equation (A2).

APPENDIX 2

For $v_0 = 0$ and $\theta_0 \neq 0$, elimination of c_3 from equation (25) and substitution of equation (3) gives:

$$1 = \frac{1}{3}\left[\frac{20Q_b^*}{Bi^2(4m+1)}\right]^{1/3}\theta_0^{(2m+3)/5}(1-\theta_0^{4m+1})^{-1/3}F\left[\frac{3}{5},\frac{22m+8}{5(4m+1)};\frac{8}{5};1-\theta_0^{4m+1}\right]. \tag{A5}$$

Using the relation [20]:

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z),$$

equation (A5) may be rewritten in the form:

$$1 = \frac{1}{3}\left[\frac{20Q_b^*}{Bi^2(4m+1)(1-\theta_0^{4m+1})}\right]^{1/3}\times F\left[1,\frac{2m}{4m+1};\frac{8}{5};1-\theta_0^{4m+1}\right]. \tag{A6}$$

From the formula:

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

the hypergeometric function in equation (A6) can be reduced to a simpler form, i.e.

$$F\left[1,\frac{2m}{4m+1};\frac{8}{5};1-\theta_0^{4m+1}\right] \simeq \frac{3(4m+1)}{2m+3} \quad \text{as } \theta_0 \to 0. \tag{A7}$$

When θ_0 approaches zero, it can be easily shown from equation (A6) with the above relationship that:

$$Bi \simeq 2(4m+1)\left[\frac{5Q_b^*}{(2m+3)^3}\right]^{1/2}. \tag{A8}$$

The result is the same as that derived in equation (35).